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DIAGONALIZATION OF MATRICES OVER RING WITH FINITE STABLE RANK

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In the present work we construct a theory of diagonalizability for matrices over rings with finite stable rank. We prove that if R is a regular ring, then every $m \times k$ and $k \times m$ matrices, where $m \geqslant bsr(R) + 2$, admits a diagonal reduction. If R is a directly finite regular ring, then R_n is directly finite for all $n \geqslant bsr(R) + 2$. We obtain an affirmative answer in greater generality to the question of Henriksen: if R is a right Bezout ring and $R/J(R)$ is a right Hermite ring, then R is right Hermite. An affirmative answer to this question implies that a commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.

Key words: stable rank, Bezout ring, elementary transformations, Hermite ring.

1. The aim of this paper is to study the question of diagonalizability for matrices over ring. In [1] Henriksen proved that if R is a unit regular ring, then every matrix over R admits diagonal reduction. The diagonalizability question for matrices was answered by Menal and Moncasi [2, Theorem 7], they showed that all matrices over regular ring R admit diagonal reductions if only if R is Hermite. Further, the stable rank (in the sense of K-theory) of a regular ring satisfying the above condition is at most 2 [2, Proposition 8].

We construct a theory of diagonalizability for matrices over rings with finite stable rank. We provide that if R is a regular ring with finite stable rank $bsr(R)$, then every $k \times m$ and $m \times k$ matrices over R, where $m \geqslant bsr(R) + 2$, admit diagonal reduction. We provide an answer to a question in [4]: if R is a directly finite regular ring, is R_n directly finite? We prove that if R is directly finite regular ring with finite stable rank $bsr(R)$, then R_n is directly finite for all $n \geqslant bsr(R) + 2$. We also obtain an affirmative answer to a question of Henriksen [6, Question 2]: if R is an right Bezout ring and $R/J(R)$ is a right Hermite ring, then R is right Hermite. An affirmative answer to this question implies that a commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.

All rings we consider are supposed to be associative with $1 \neq 0$. By a right Bezout ring we will mean a ring in which all finitely generated right ideals are principal, and by a Bezout ring a ring which is both right and left Bezout. We recall that a module is uniserial if its lattice of submodules forms a chain. A ring is right serial if as a right module over itself, it is a direct sum of uniserial modules. A ring is serial if it both right ad left serial [5].

We shall call two matrices A and B over a ring R equivalent, if there exist invertible matrices P,Q such that $B = PAQ$. An matrix A admits diagonal reduction if A is equivalent to a diagonal matrix. If every $1 \times n$ ($n \times 1$) matrix over R admits diagonal reduction, then R is n-right (left) Hermite. A right (left) Hermite ring is a ring which is n-right (left) Hermite, for any $n \geqslant 1$. A ring which is both right and left Hermite is an Hermite ring. Obviously a right Hermite ring is right Bezout. A ring R is said to be regular if for every $a \in R$ there exists $x \in R$ such that $axa = a$. It is easy to see that a regular ring is Bezout [4]. A row $(a_1, \dots, a_n)$ over a ring R is called right unimodular, if $a_1R + \cdots + a_nR = R$. If $(a_1, \dots, a_n)$ is a right unimodular n-row over a ring R, then we say that $(a_1, \dots, a_n)$ is reducible if there exists an $(n-1)$-row $(b_1, \dots, b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1, \dots, a_{n-1} + a_nb_{n-1})$ is a right unimodular $(n-1)$-row. A ring R is said to have stable rank $n \geqslant 1$, if n is the least positive integer such that every right unimodular $(n+1)$-row is reducible. This number is denoted by $bsr(R)$. A ring R is directly finite if $xy = 1$ implies $yx = 1$ for all $x, y \in R$.

We denote by R_n the ring of all $n \times n$ matrices over R, and by $GL_n(R)$ its group of unities. We write $GE_n(R)$ for the subgroup of $GL_n(R)$ generated by elementary matrices. The Jacobson radical of a ring R will be denoted by $J(R)$. Denote by $U(R)$ the group of unities of R.

2. Diagonalization of matrices over ring with finite stable rank.

Proposition 1. *Let R be a right Bezout ring with finite stable rank $bsr(R)$. Then any right unimodular row of length m over R, where $m \geqslant bsr(R)+1$, can be completed to an invertible matrix in $GE_m(R)$.*

Proof. If $a_1R + \cdots + a_{n+1}R = R$, then there exists an m-row $(c_1, \dots, c_m)$ with

$$(a_1 + a_{m+1}c_1)R + \cdots + (a_m + a_{m+1}c_m)R = R.$$

There exist $u_1, \dots, u_m \in R$ such that

$$(a_1 + a_{m+1}c_1)u_1 + \cdots + (a_m + a_{m+1}c_m)u_m = 1.$$

Set

$$P_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \dots & c_m & 1 \end{pmatrix} \in GE_{m+1}(R),$$

$$P_2 = \begin{pmatrix} 1 & 0 & \dots & 0 & u_1(1-a_{m+1}) \\ 0 & 1 & \dots & 0 & u_2(1-a_{m+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & u_m(1-a_{m+1}) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in GE_{m+1}(R).$$

We see that for a row $(a_1, \dots, a_{m+1})P_1P_2$ there exists a matrix $P_3 \in GE_{m+1}(R)$ such that $(a_1, \dots, a_{m+1})P_1P_2P_3 = (1, 0, \dots, 0)$. Thus we obtain a matrix $P \in GE_{m+1}(R)$ such that $(a_1, \dots, a_{m+1})P = (1, 0, \dots, 0)$. Then $(a_1, \dots, a_{m+1})$ is the first row of the matrix P^{-1}. For any right unimodular row of length $> m+1$ the result follows by induction.

Proposition 2. *Let R be a right Bezout ring with finite stable rank $bsr(R)$, then R is an m-right Hermite ring, for any $m \geqslant bsr(R)+1$.*

Proof. Since R is a right Bezout ring, then for any $a_1, \dots, a_m \in R$ there exists $d \in R$ such that $a_1R + \dots + a_mR = dR$. Say $a_1u_1 + \dots + a_mu_m = d$, $a_1 = db_1$, $\dots$, $a_m = db_m$. From these relations we get $d(b_1u_1 + \dots + b_mu_m - 1) = 0$ so that $b_1R + \dots + b_mR + cR = R$ for some $c \in R$ such that $dc = 0$. Since $m \geqslant bsr(R)+1$, we have $(b_1 + cx_1)R + \dots + (b_m + cx_m)R = R$, where $x_1, \dots, x_n \in R$. By Proposition 1, we can find an invertible matrix $P \in GE_m(R)$ of the form

$$P = \begin{pmatrix} b_1 + cx_1, & \dots, & b_m + cx_m \\ & * & \end{pmatrix}.$$

Clearly $(a_1, \dots, a_m)P^{-1} = (d, 0, \dots, 0)$, some R is m-right Hermite.

Now we are ready to prove a result which characterizes the regular rings which have finite stable rank.

Theorem 1. *Let R be a regular ring with finite stable rank $bsr(R)$. Then for every $k \times m$ $(m \times k)$ matrices A over R, where $m \geqslant bsr(R)+2$, there exist invertible matrices $P \in GE_k(R)$ $(P \in GE_m(R))$, $Q \in GE_m(R)$ $(Q \in GE_k(R))$ such that PAQ is a diagonal matrix.*

Proof. In order to prove that A admits diagonal reduction, we proceed by induction on k. If $k = 1$, the result follows by Proposition 2. If $k > 1$ it follows similarly as the proof of Theorem 9 [2].

Thus we provide an answer to Henriksen's question [1], whether a regular ring can be an elementary divisor ring without being unit regular.

Theorem 2. *Let R be a directly finite ring. If every $n \times n$ matrix over R is equivalent to a diagonal matrix, then R_n is a directly finite ring.*

Proof. Let $A, B \in R_n$ and $AB = E$, the identity n-matrix. If

$$PAQ = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix} = \varepsilon,$$

where $P, Q \in GL_n(R)$, then $PAQQ^{-1}BP^{-1} = \varepsilon Q^{-1}BP^{-1} = E$. Since R is directly finite, we see that $\Phi = Q^{-1}BP^{-1}$ is a diagonal matrix. Since R is directly finite, we obtain $\Phi\varepsilon = \varepsilon\Phi = E$ and $\varepsilon \in GL_n(R)$. Thus $A = P^{-1}\varepsilon Q^{-1} \in GL_n(R)$ and $BA = E$ and hence R_n is directly finite.

Theorem 3. *Let R be a directly finite regular ring with finite stable rank $bsr(R)$. Then R_m is directly finite for every $m \geqslant bsr(R)+2$.*

This theorem follows from Theorem 1 and Theorem 2.

Theorem 2.5 in [3] provides a large class of regular rings over which all square matrices are diagonalizable, these rings are separative regular rings. Then we have

Theorem 4. *Let R be directly finite separative regular ring. Then R_n is directly finite for all n.*

Levy in [5] proved that all square matrices over serial rings are diagonalizable. Then we have

Theorem 5. *Let R be a directly finite serial ring. Then R_n is directly finite for all n.*

We obtain an affirmative answer to a question of Henriksen [6, Question 2].

Theorem 6. *Let R be a right Bezout ring, and $R/J(R)$ is a right Hermite ring. Then R is right Hermite.*

Proof. We show first that any right unimodular row over R can be completed to an invertible matrix. Set $\overline{R} = R/J(R)$. Let $aR + bR = R$, then $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$. Since $\overline{R}$ is a right Hermite ring, the right unimodular row $(\overline{a}, \overline{b})$ over $\overline{R}$ can be completed to an invertible matrix

$$\overline{A} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{u} & \overline{v} \end{pmatrix}.$$

Thus $\overline{AC} = \overline{CA} = \overline{E}$. Let

$$\overline{C} = \begin{pmatrix} \overline{c} & \overline{x} \\ \overline{d} & \overline{y} \end{pmatrix}.$$

Then $ac + bd = 1 + j_1$, $ax + by = j_2$, $uc + vd = j_3$, $ux + vy = 1 + j_4$, for any $j_1, j_2, j_3, j_4 \in J(R)$. Set

$$A = \begin{pmatrix} a & b \\ u & v \end{pmatrix}, \quad C = \begin{pmatrix} c & x \\ d & y \end{pmatrix},$$

then

$$AC = \begin{pmatrix} 1 + j_1 & j_2 \\ j_3 & 1 + j_4 \end{pmatrix} = J.$$

Since $1 + j_1 \in U(R)$, then $J \in GL_2(R)$ and $A \in GL_2(R)$.

Now we prove that R is right Hermite ring. Suppose that we are given $a, b \in R$, then $aR + bR = dR$, say $a = da_0$, $b = db_0$, $d = au + bv$. From these relations we get $d(a_0u + b_0v - 1) = 0$, so $a_0R + b_0R + c_0R = R$ for some $c_0 \in R$ such that $dc_0 = 0$. Since $\overline{R}$ is a right Hermite ring, then $bsr(\overline{R}) \leqslant 2$ [2, Proposition 8]. Since for the ring R the following assertion hold: $u \in U(R)$ if and only if $u + J(R) \in U(\overline{R})$, then $bsr(R) \leqslant 2$. Thus $(a_0 + c_0x)R + (b_0 + c_0y)R = R$, where $x, y \in R$. By the above argument, we can find an invertible matrix of the form

$$P = \begin{pmatrix} a_0 + c_0x & b_0 + c_0y \\ * & * \end{pmatrix}.$$

Clearly $(a, b)P^{-1} = (d, 0)$, so R is right Hermite.

Theorem 7. *A commutative Bezout ring is an elementary divisor ring if and only if $R/J(R)$ is an elementary divisor ring.*

Proof. Obviously, every homomorphic image of an elementary divisor ring is an elementary divisor ring, so we have only to prove the sufficiency. Let $R/J(R)$ be an elementary divisor ring, then by Theorem 6, R is Hermite. By [6, Theorem 3] R is an elementary divisor ring.

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ДІАГОНАЛІЗАЦІЯ МАТРИЦЬ НАД КІЛЬЦЯМИ СКІНЧЕННОГО СТАБІЛЬНОГО РАНГУ

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Побудовано теорію діагоналізації матриць над кільцями скінченного стабільного рангу. Доведено таке: якщо R – регулярне кільце, то довільні $m \times k$ і $k \times m$ матриці над R, де $m \geqslant ст.р.(R) + 2$, володіють діагональною редукцією. Якщо R прямо скінченне регулярне кільце, то кільце матриць R_n є прямо скінченне для довільного $n \geqslant ст.р.(R) + 2$. Показано таке: якщо R праве кільце Безу таке, що $R/J(R)$ є правим кільцем Ерміта, тоді R праве кільце Ерміта. Одержали, що комутативне кільце Безу є кільцем елементарних дільників тоді і тільки тоді, коли $R/J(R)$ кільце елементарних дільників.

Ключові слова: стабільний ранг, кільце Безу, елементарна редукція, кільце Ерміта.

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